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LETTER TO THE EDITOR

Study of the 1D-biased and 2D ultradiffusion models

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Abstract. In this letter we study, within a renormalisation group scheme, two possible generalisations of a one-dimensional (1D) diffusion model with a hierarchy of energy barriers recently proposed. We consider the same 1D model plus a biasing term and a 2D version of it. In the first case we find, for a suitable value of the biasing force, a new transition in the long-time behaviour of the autocorrelation function. For the 2D model, through an approximate treatment, we obtain the corresponding anomalous exponent which characterises its relaxational behaviour.

Recently, Huberman and Kerszberg (1985) (HK) studied a model of diffusion on a line with energy barriers distributed in a hierarchical way, as shown schematically in figure 1. There, the vertical segments represent barriers which must be hopped by a particle to move from a cell to its nearest neighbours. The length of each segment is inversely proportional to the probability ε_i for the corresponding barrier to be crossed in a unit of time. As these authors pointed out, the behaviour of such a system can be related to those of different real systems in which many timescales are present. These systems range from macromolecules (Austin *et al* 1975) to spin glasses (Sompolinsky 1981), strongly interacting glassy materials (Palmer *et al* 1984) and computing structures (Huberman and Hogg 1984).

Owing to the characteristic ultrametric topology of hierarchical systems, HK termed this process ultradiffusion; they also showed that its relaxation displays an anomalous decay behaviour. Through an approximate renormalisation group (RG) technique, based on the assumption of a small ratio $R=\varepsilon_{i+1}/\varepsilon_i$ of two successive barrier heights $1/\varepsilon_i$ and $1/\varepsilon_{i+1}$, they found for the autocorrelation function $P_0(t)$ a scaling behaviour $P_0(t)\sim t^{-x/2}$ $(t\to\infty)$, with the non-universal exponent x given by

$$x=2\ln 2/\ln(2/R) \qquad (R\simeq 0). \tag{1}$$

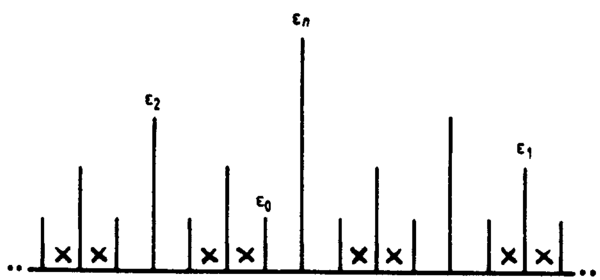

Figure 1. Hierarchical array of barriers of the 1D ultradiffusion model. The cells marked with crosses are decimated in the RG procedure.

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More recently, Teitel and Domany (1985) (TD) showed that this anomalous behaviour extends up to $R_c = \frac{1}{2}$. At this value a phase transition to normal relaxation ($x = 1$) takes place, and ordinary diffusion is present up to $R = 1$. This result was obtained through approximated decimation calculations for $R \simeq 0$ and for $R \simeq 1$, which were complemented with numerical methods.

At the same time, Maritan and Stella (1985) (MS) performed an exact RG treatment of the model under consideration. They obtained

$$x = 2\ln 2/\ln\left(\frac{2(\varepsilon_0 + 2\varepsilon_1^*)}{\varepsilon_1^*}\right) \tag{2}$$

where ε_1^* characterises the line of fixed points to which the initial barrier hierarchy $\{\varepsilon_n\}$ is attracted. For ε_1^* ranging from 0 to ∞ this equation gives a continuously varying exponent, interpolating between trapping ($x = 0$) and normal diffusion ($x = 1$).

The exact result (2) was obtained without any supposition on the values of the transition rates ε_i. However, it must be stressed that for the particular hierarchy considered by HK and TD, and by using the recursion relations found by MS, one obtains

$$\varepsilon_1^* = \frac{R}{(1-2R)}\varepsilon_0. \tag{3}$$

Thus, for $0 \leq R < \frac{1}{2}$ the result (2) coincides with (1); when $\frac{1}{2} \leq R \leq 1$ equation (3) shows the breakdown of the MS analysis. So the approach adopted by these authors is incapable of describing the phase transition found by TD.

It is interesting to point out that the results (1)-(3) are valid independent of the n_0 from which the relation $\varepsilon_{n+1} = R\varepsilon_n$ begins to hold, and regardless of the values ε_0, $\varepsilon_1, \ldots, \varepsilon_{n_0}$.

In this letter we consider two possible generalisations of the model discussed above. First, we study the one-dimensional (1D) model, with arbitrary transition rates ε_i, in the presence of an external force which biases the diffusion in a privileged direction. For the biasing term of (4) and with $\min\{\varepsilon_n\} < \varepsilon_0$ we found that the behaviour (2) of the autocorrelation function is not modified by the bias. However, if $\min\{\varepsilon_n\} = \varepsilon_0$ there is a transition at $\eta = \varepsilon_0$, from power law to (hierarchy-dependent) exponential decay of $P_0(t)$ (equation (6)).

Second, we study a 2D version of the unbiased model, with the same distribution of barriers as shown in figure 1 on each direction. For this particular hierarchy—which allows for an approximate calculation in much the same way as in HK—we found an exponent x which coincides with advanced results (TD).

As usual, we denote by $P_n(t)$ the probability of finding the particle at cell n at time t, and $\tilde{P}_n(\omega)$ its Laplace transform. Then, for the 1D-biased system we have the following equations:

$$\begin{aligned}\omega\tilde{P}_n &= \delta_{n,0} + (\varepsilon_{n-1,n} + \eta)\tilde{P}_{n-1} + (\varepsilon_{n+1,n} - \eta)\tilde{P}_{n+1} - (\varepsilon_{n,n-1} + \varepsilon_{n,n+1})\tilde{P}_n \\ &= \delta_{n,0} + \varepsilon_{n-1,n}(\tilde{P}_{n-1} - \tilde{P}_n) + \varepsilon_{n,n+1}(\tilde{P}_{n+1} - \tilde{P}_n) + \eta(\tilde{P}_{n-1} - \tilde{P}_{n+1})\end{aligned} \tag{4}$$

where the last term on the second line is the biasing one. In these equations $\delta_{n,0}$ means that the particle starts its diffusion from the cell 0, and $\varepsilon_{n,m}$ is the corresponding hopping rate between neighbouring cells n and m. From the first line of (4) one can see that the model makes sense only for $\eta \leq \min\{\varepsilon_n\}$.

Following MS, we perform an exact decimation which eliminates the cells with crosses in figure 1†. This leads to a new system of the same form as (4) with the following rescaling:

$$\varepsilon'_{n-1}=\Delta_1\varepsilon_n \qquad \eta'=\Delta_1\eta$$
$$\tilde{P}'_n=\tilde{P}_n/\Delta_1 \qquad \omega'=\Omega\omega \tag{5a}$$

where

$$\Delta_1=\frac{\varepsilon_0^2\varepsilon_1+\eta^2(\varepsilon_1+2\varepsilon_0)}{\varepsilon_0(\varepsilon_0^2+2\varepsilon_0\varepsilon_1+\eta^2)}$$
$$\Omega=\frac{2\varepsilon_0[\varepsilon_0^2+2\varepsilon_0\varepsilon_1+\eta(\varepsilon_0+\varepsilon_1)+\eta^2]}{\varepsilon_0^2\varepsilon_1+\eta^2(\varepsilon_1+2\varepsilon_0)}. \tag{5b}$$

In deriving these recursion relations we have removed terms of $O(\omega)$ in Δ_1 and Ω in order to obtain the dominant singular behaviour of $\tilde{P}_0(\omega)$ for $\omega\to 0$ $(t\to\infty)$. It has also been imposed that $\varepsilon'_0=\varepsilon_0$, which fixes the unit of time.

The recursion $\eta'=\Delta_1\eta$ plus the restriction $\eta\leqslant\min\{\varepsilon_n\}$ has the following fixed points:

(i) if $\min\{\varepsilon_n\}<\varepsilon_0$ the only fixed point is $\eta^*=0$, which corresponds to the unbiased case discussed by MS and TD;

(ii) if $\min\{\varepsilon_n\}=\varepsilon_0$ we have, in addition to $\eta^*=0$, the new fixed point $\eta^*=\varepsilon_0$. For this value of η the system of equations (4) decouples in small subsystems and then it can be fully solved.

Supposing the cell $n=0$ to be the left-hand one marked with a cross in figure 1, we obtain

$$P_0(t)=\tfrac{1}{2}(\mathrm{e}^{-\lambda_1 t}+\mathrm{e}^{-\lambda_2 t}) \tag{6a}$$

where

$$\lambda_{1,2}=(\varepsilon_1+\varepsilon_0)\pm(\varepsilon_1^2-\varepsilon_0^2)^{1/2}. \tag{6b}$$

Hence, in this case we have, in addition to the transition found by TD, a new transition in the long-time behaviour of the autocorrelation function, from the power law $P_0(t)\sim t^{-x/2}$ with x given by (2) for $\eta^*=0$ to the simple exponential decay (6) for $\eta^*=\varepsilon_0$. Note that this exponential decay is dependent on the particular values of ε_0 and ε_1.

For the 2D model we consider the following barrier hierarchy:

$$\varepsilon_0 \qquad \varepsilon_1 \qquad \varepsilon_n=\varepsilon_1 R^{n-1} \qquad (n\geqslant 2) \tag{7}$$

with arbitrary ε_0, ε_1 and $R\ll 1$. The method used is based on the decimation procedure illustrated in figure 2, which generalises that employed in the 1D case.

Decimating the shaded cells of figure 2(a) we obtain—after a long calculation which involves handling the sixteen equations for the cells in this figure—the new system of figure 2(b). This new system is characterised by the following rescaled quantities:

$$\varepsilon'_n=\Delta_2\varepsilon_{n+1} \tag{8a}$$
$$\tilde{P}'_n=\tilde{P}_n/\Delta_2 \qquad \omega'=4\Delta_2\omega \tag{8b}$$

† TD decimate odd sites in the chain and write an effective master equation for the remaining even sites. Such a decimation is only valid up to $O(R)$ or $O(1-R)$; at higher orders it produces a non-hierarchical array of barriers.

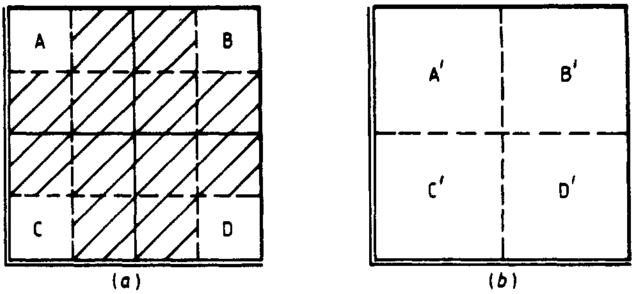

Figure 2. (*a*) Two-dimensional array of barriers. The hopping rates are: - - - ≡ ε_0, —— ≡ ε_1, —— ≡ ε_2 and ═══ ≡ ε_n. The shaded cells are decimated in the approximate RG procedure. (*b*) Array of barriers after the decimation with hopping rates: - - - ≡ ε_0', —— ≡ ε_1' and ═══ ≡ ε_{n-1}'.

where

$$\Delta_2 = \frac{\varepsilon_0^2 + 6\varepsilon_0\varepsilon_1 + 4\varepsilon_1^2}{2\varepsilon_1(\varepsilon_0 + \varepsilon_1)}.$$

In this case to obtain the above approximate recursion relations we have also removed terms of $O(R)$. Moreover, we have imposed that (7) retain its form with $R' = R$ and $\varepsilon_0' = \varepsilon_0$. Note that the two-parameter hierarchy $\varepsilon_n = \varepsilon_0 R^n$ $(n = 1, 2, \ldots)$ is not closed under the RG transformations (8). By replacing ε_n' and ε_{n+1} in terms of R' and R in (8*a*), a recursion relation for R which depends on n is obtained.

The recursions (8*a*) have a line of fixed points defined by

$$\varepsilon_1^* = \varepsilon_0 R/2 \qquad \varepsilon_n^* = \varepsilon_1^* R^{n-1} \qquad (R \simeq 0).$$

The relations (8*b*) are consistent with a potential decay of the autocorrelation function. The decay exponent, obtained by antitransforming $P_0(\omega, \varepsilon_1^*)$, is

$$x_{2D} = 4 \ln 2 / \ln(2\varepsilon_1^*/\varepsilon_0).$$

This result should be compared with the 1D analogous exact result (2) when $\varepsilon_1^* \ll \varepsilon_0$:

$$x_{1D} = 2 \ln 2 / \ln(2\varepsilon_1^*/\varepsilon_0)$$

which suggests the relation $x_{dD} = dx_{1D}$ for a general hierarchy structure in d dimensions. This relation has been advanced for the hierarchy $\varepsilon_n = \varepsilon_0 R^n$ by TD.

In conclusion, we have shown that a biasing term in the 1D ultradiffusion model can produce, for a suitable value of the biasing force, a transition in the relaxational behaviour of the autocorrelation function, from potential to (hierarchy-dependent) exponential decay. Also, we have found an approximate expression for the anomalous exponent corresponding to the relaxational behaviour of a 2D system. From this result we suggest a general expression for the exponent associated with a d-dimensional ultradiffusion process with arbitrary hierarchy structure.

We would like to thank the first referee for bringing the paper by TD to our attention.

References

Austin R H, Berson K W, Eisenstein L, Frauenfelder L H and Gunsalus I C 1975 *Biochem.* **14** 5355
Huberman B A and Hogg T 1984 *Phys. Rev. Lett.* **52** 1048
Huberman B A and Kerszberg M 1985 *J. Phys. A: Math. Gen.* **18** 1331
Maritan A and Stella A 1985 unpublished
Palmer R G, Stein D L, Abrahams E and Anderson P W 1984 *Phys. Rev. Lett.* **53** 958
Sompolinsky H 1981 *Phys. Rev. Lett.* **47** 935
Teitel S and Domany E 1985 *Phys. Rev. Lett.* **55** 2176